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Note

Average independence polynomials

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Abstract

The *independence polynomial* of a graph G is the function $i(G,x) = \sum_{k \geqslant 0} i_k x^k$, where i_k is the number of independent sets of vertices in G of cardinality k. We investigate here the *average* independence polynomial, where the average is taken over all independence polynomials of (labeled) graphs of order n. We prove that while almost every independence polynomial has a nonreal root, the average independence polynomials always have all real, simple roots.

Keywords: Graph; Independence; Polynomial; Roots

1. Introduction

We let $\mathcal{G}_n$ denote the set of all simple labelled graphs on $\{1, \ldots, n\}$, and for a graph G, $\mathcal{I}(G)$ denote the set of all independent sets of G. For a graph G with independence number β, let i_k denote the number of independent sets of vertices of cardinality k in G ($k = 0, 1, \ldots, \beta$); the *independence polynomial* of G,

$$i(G,x) = \sum_{k=0}^{\beta} i_k x^k$$

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is the generating polynomial for the sequence. There has been considerable interest in independence polynomials and their roots (c.f. [3–6,8–10]). We are interested in the average of the independence polynomials over all simple graphs of order n:

$$\mathrm{aip}_n(x) = \frac{1}{2^{\binom{n}{2}}} \sum_{G \in \mathcal{G}_n} i(G, x).$$

(We remark that other average graph polynomials have been investigated by others. In particular, the average chromatic polynomial was previously introduced by Welsh [12].) We shall find closed forms for these polynomials, and investigate the nature of the roots.

As a point of contrast, the *average matching polynomial* is

$$\mathrm{amp}_n(x) = \frac{1}{2^{\binom{n}{2}}} \sum_{G \in \mathcal{G}_n} \mu(G, x),$$

where $\mu(G, x)$ is the well-known *matching polynomial* of G:

$$\mu(G, x) = \sum_{k \geqslant 0} (-1)^k m_i x^{n-2k},$$

where m_i is the number of matchings of G of size i. It is not hard to verify that

$$\mathrm{amp}_n(x) = \mu(K_n, 2x)$$

so that the average matching polynomial has all real roots (as the matching polynomials of complete graphs, which form the orthogonal family of Hermite polynomials, have all real roots (c.f. [7])). It may not be surprising that the average matching polynomial has all real roots as *every* matching polynomial has all real roots (c.f. [7]). On the other hand, the average independence polynomials are not as easily expressible, and we shall see that most independence polynomials have nonreal roots. In this context, that the average independence polynomials have all real roots is unexpected.

We begin by first showing that almost all independence polynomials have a nonreal root (in contrast, the independence polynomials of line graphs, being easily expressed into terms of matching polynomials, have all real roots).

Proposition 1. *With probability tending to* 1, *the independence polynomial of G has a nonreal root.*

Proof. Note that for a graph G with independence number β, $I(G, x)$ has a nonreal root iff $g(G, x) = x^{\beta} i(G, 1/x)$ has a nonreal root. The latter is $x^{\beta} + nx^{\beta-1} + mx^{\beta-2} + tx^{\beta-3} + \cdots$, where n, m and t are the number of vertices, edges and triangles in the complement of G. It suffices to show that (with probability tending to 1)

$$f(G, x) = x^{n-\beta} g(G, x) = x^n + nx^{n-1} + mx^{n-2} + tx^{n-3} + \cdots$$

has a nonreal root. Now it easily follows from Rolle's theorem if a polynomial has all real roots, then so does its derivative. Hence it suffices to show that $f(G, x)$ has a nonreal root.

We consider the Sturm sequence of f (the first term is the original polynomial, the second is its derivative, and subsequent ones are the negative of the remainder when dividing the previous one by the one before that one—the sequence ends when the next term is the zero polynomial):

$$\begin{aligned} f_0 &= f \\ &= x^n + nx^{n-1} + mx^{n-2} + tx^{n-3} + \cdots, \\ f_1 &= nx^{n-1} + n(n-1)x^{n-2} + m(n-2)x^{n-3} + t(n-3)x^{n-4} + \cdots, \\ f_2 &= \frac{n(n-1)-2m}{n}x^{n-2} + \frac{m(n-2)-3t}{n}x^{n-3} + \cdots. \end{aligned}$$

Sturm's Theorem (c.f. [11]) states that a polynomial has all real roots iff all the leading coefficients in its Sturm sequence have the same sign and that the degrees drop by exactly one (though there can be a larger drop when the next polynomial is the zero polynomial). Clearly f_0 and f_1 have positive leading coefficient and (unless the graph is complete, which happens with probability tending to 0), f_2 also has positive leading coefficient.

However, we calculate f_3, the next term in the Sturm sequence of f. It begins as

$$\frac{n^3m^2 - 2n^2m^2 - 4nm^3 + 8m^3 - 3n^4t + 6n^3t + 12mn^2 - 3n^2t - 18mnt - 9nt^2}{(n^2-n-2m)^2} \times x^{n-4} + \cdots .$$

The denominator is positive, so if we can show that the numerator N of the leading coefficient of f_3 is negative, we conclude that f has a nonreal root and we are done.

The distribution of m is binomial with parameters $\binom{n}{2}$ and $p = 1/2$, and hence by [2, p. 12], for any fixed $\varepsilon_1 \in (0, 1)$,

$$\text{Prob}(|m - n(n-1)/4| \leqslant \varepsilon n) \geqslant 1 - 2e^{-2\varepsilon_1^2 n} \to 1.$$

For the distribution of t, we take an indicator random variable X_s for the event A_s of a triangle s occurring, so that $t = \sum X_s$. It is easy to see that

$$E(t) = (1/8)\binom{n}{3} \to \infty.$$

As in [1, p. 40], set

$$\varDelta = \sum_{s\sim r} \text{Prob}(A_s \text{ and } A_r),$$

where $s \sim r$ iff $s \neq r$ and the events A_s and A_r are not independent, that is, iff s and r share an edge (the sum is over all ordered pairs of such s and r). Then an easy calculation shows that

$$\varDelta = \binom{n}{2}(n-2)(n-3)/32 = o((E(t))^2)$$

and hence by [1, Corollary 3.4] we have that for any $\varepsilon_2 \in (0, 1)$,

$$\text{Prob}\left(\left|t - (1/8)\binom{n}{3}\right| \leqslant \varepsilon_2 (1/8)\binom{n}{3}\right) \to 1.$$

We can assume now that (with probability tending to 1) that $m = \varepsilon_1 n(n-1)/4$ and $t = \varepsilon_2 n(n-1)(n-2)/48$. Substituting this into N yields a seventh degree polynomial with leading coefficient

$$\frac{\varepsilon_1\varepsilon_2}{16} + \frac{\varepsilon_1^2}{16} - \frac{\varepsilon_2}{16} - \frac{\varepsilon_1^3}{16} - \frac{\varepsilon_2^2}{256}.$$

The latter, as a function of ε_1 and ε_2, has no critical points in $[0.99, 1.01] \times [0.99, 1.01]$, and in fact, its maximum on this box is less than -0.000382. Hence for $(\varepsilon_1, \varepsilon_2) \in [0.99, 1.01] \times [0.99, 1.01]$, N is a polynomial with negative leading coefficient, and hence negative for large n (the other coefficients are bounded on this box as well). Hence f_3 has negative leading coefficient for n sufficiently large, and so we conclude that the independence polynomial of almost every graph has a nonreal root. $\square$

While almost every independence polynomial has a nonreal root, we shall see that the average independence polynomial is much better behaved. To do so, however, we shall need to find an explicit formula for $\text{aip}_n(x)$.

Note that the sum

$$\begin{aligned}\sum_{G\in\mathcal{G}_n} i(G,x) &= \sum_{G\in\mathcal{G}_n}\sum_{I\in\mathcal{I}(G)} x^{|I|}\\ &= \sum_{I\subseteq\{1,\ldots,n\}}\ \sum_{G\in\mathcal{G}_n \text{ with } I\in\mathcal{I}(G)} x^{|I|}\\ &= \sum_{k=0}^{n}\binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}} x^k\end{aligned}$$

so it follows that

$$\text{aip}_n(x) = \sum_{k=0}^{n}\frac{\binom{n}{k}}{2^{\binom{k}{2}}} x^k.$$

Theorem 2. *The average independence polynomial has all real, simple roots.*

Proof. We show by induction that $\text{aip}_n(x)$ has roots $r_{n,n} < r_{n,n-1} < \cdots < r_{n,1} < 0$ such that $r_{n,i+1} < 2r_{n,i}$ for $i = 1, 2, \ldots, n-1$. The result is clear for $n = 1$, as $\text{aip}_1(x) = 1 + x$ has $r_{1,1} = -1$. Moreover, all real roots of any $\text{aip}_n(x)$ must be negative, as these polynomials have all positive coefficients.

Now assume that $\text{aip}_{n-1}(x)$ has roots $r_{n-1,n-1} < r_{n-1,n-2} < \cdots < r_{n-1,1} < 0$ such that $r_{n-1,i+1} < 2r_{n-1,i}$ for $i = 1, 2, \ldots, n-2$. Note that the signs of $\text{aip}_{n-1}(x)$ on the intervals $(-\infty, r_{n-1,n-1})$, $(r_{n-1,n-1}, r_{n-1,n-2}), \ldots, (r_{n-1,2}, r_{n-1,1})$, $(r_{n-1,1}, 0)$ are $(-1)^{n-1}$, $(-1)^{n-2}, \ldots, -1, 1$.

From (1) we derive that

$$\text{aip}_n(x) = \text{aip}_{n-1}(x) + x\,\text{aip}_{n-1}(x/2)$$

since we have

$$\begin{aligned}
\mathrm{aip}_n(x) &= \sum_{k=0}^{n} \frac{\binom{n}{k}}{2^{\binom{k}{2}}} x^k \\
&= \sum_{k=0}^{n} \frac{\binom{n-1}{k} + \binom{n-1}{k-1}}{2^{\binom{k}{2}}} x^k \\
&= \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2^{\binom{k}{2}}} x^k + x \sum_{k=1}^{n} \frac{\binom{n-1}{k-1}}{2^{\binom{k}{2}}} x^{k-1} \\
&= \mathrm{aip}_{n-1}(x) + x \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2^{\binom{k+1}{2}}} x^k \\
&= \mathrm{aip}_{n-1}(x) + x \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2^k \cdot 2^{\binom{k}{2}}} x^k \\
&= \mathrm{aip}_{n-1}(x) + x\, \mathrm{aip}_{n-1}(x/2).
\end{aligned}$$

Now we see from (1) that

$$\begin{aligned}
\mathrm{sign}(\mathrm{aip}_n(r_i)) &= -\mathrm{sign}(\mathrm{aip}_{n-1}(r_i/2)) \\
&= -(-1)^{i+1} \\
&= (-1)^i
\end{aligned}$$

as $r_i < r_i/2 < r_{i-1}$ by our inductive hypothesis. Moreover,

$$\begin{aligned}
\mathrm{sign}(\mathrm{aip}_n(2r_i)) &= \mathrm{sign}(\mathrm{aip}_{n-1}(2r_i)) \\
&= (-1)^i
\end{aligned}$$

as $r_{i+1} < 2r_i < r_i$ by our inductive hypothesis. It follows that $\mathrm{aip}_n(x)$ changes sign at $2r_{n-1,i}$ and $r_{n-1,i+1}$ for $i = 1, \ldots, n-2$. Moreover, $\mathrm{aip}_n(r_{n-1,1})$ has sign -1 while $\mathrm{aip}_n(0)$ has sign 1, and $\mathrm{aip}_n(2r_{n-1,n-1})$ has sign $(-1)^{n-1}$ while $\mathrm{aip}_n(x)$ has sign $(-1)^n$ as x tends to $-\infty$. We conclude by the intermediate value theorem that $\mathrm{aip}_n(x)$ has roots $\gamma_n, \ldots, \gamma_1$ in each of the intervals $(-\infty, 2r_{n-1,n-1}), (r_{n-1,n-1}, 2r_{n-1,n-2}), \ldots, (r_{n-1,2}, 2r_{n-1,1})$ and $(r_{n-1,1}, 0)$, respectively. Note that this implies that $\gamma_{i+1} > 2\gamma_i$ for $i = 1, \ldots, n-1$, so we are done. □

The proof above actually shows that the roots of $\mathrm{aip}_n(x)$ interlace the roots of $\mathrm{aip}_{n-1}(x)$.

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